NATIONAL ADVISORY COMMITTEE

FOR AERONAUTICS

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TECHNICAL NOTE

No. 961

THE "LIMITING LINE" IN MIXED SUBSONIC AND

SUPERSONIC FLOW OF COMPRESSIBLE FLUIDS

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Washington
November 1944

RESTRICTED

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THE "LIMITING LINE" IN MIXED SUBSONIC AND
SUPERSONIC FLOW OF COMPRESSIBLE FLUIDS

By Hsue-shen Tsien

It is well known that the vorticity for any fluid element is constant if the fluid is non-viscous and the change of state of the fluid is isentropic. When a solid body is placed in a uniform stream, the flow far ahead of the body is irrotational. Then if the flow is further assumed to be isentropic, the vorticity will be zero over the whole field of flow. In other words, the flow is irrotational. For such flow over a solid body, it is shown by Theodorsen (reference 1) that the solid body experiences no resistance. If the fluid has a small viscosity, its effect will be limited in the boundary layer over the solid body and the body will have a drag due to the skin friction. This type of essentially isentropic irrotational flow is generally observed for a streamlined body placed in a uniform stream, if the velocity of the stream is kept below the so-called "critical speed."

At the critical speed or rather at a certain value of the ratio of the velocity of the undisturbed flow and the corresponding velocity of sound, shock waves appear. This phenomenon is called the "compressibility burble." Along a shock wave, the change of state of the fluid is no longer isentropic, although still adiabatic. This results in an increase in entropy of the fluid and generally introduces vorticity in an originally irrotational flow. The increase in entropy of the fluid is, of course, the consequence of changing part of the mechanical energy into heat energy. In other words, the part of fluid affected by the shock wave has a reduced mechanical energy. Therefore, with the appearance of shock waves, the wake of the streamline body is very much widened, and the drag increases drastically. Furthermore, the accompanying change in the pressure distribution over the body changes the aerodynamic moment acting on it and in the case of an airfoil decreases the lift force.

All these consequences of the breakdown of isentropic irrotational flow are generally undesirable in applied aerodynamics. Its occurrence should be delayed as much as possible by modifying the shape or contour of the body. However, such endeavor will be very much facilitated if the cause or the criterion for the breakdown can be found first.

CRITERION FOR THE BREAKDOWN OF

ISENTROPIC IRROTATIONAL FLOW

Taylor and Sharman (reference 2) calculated the successive approximations to the flow around an airfoil by means of an electrolytic tank. They found that when the maximum velocity in the flow reaches the local velocity of sound, the convergence of the successive steps seems to break down. This fact led to the identification of critical speed or critical Mach number with the Mach number of the undisturbed flow for which the local velocity at some point reaches the local velocity of sound. However, there is no mathematical proof for the coincidence of the critical Mach number so defined and the breakdown of isentropic irrotational flow. Furthermore, such a definition for critical Mach number implies that a transition from a velocity less than that of sound, or subsonic velocity, to a velocity greater than that of sound, or supersonic velocity, does not occur in isentropic irrotational flow. On the other hand, Taylor (reference 3) and others found solutions for which such a transition occurs. Furthermore, Binnie and Hooker (reference 4) have shown that at least for the case of spiral flow the method of successive approximation is a convergent one even for supersonic velocities. With these facts in mind, it may be concluded that the identification of critical speed with local supersonic velocity cannot be correct.

Taylor's investigation on the spiral flow (reference 3) indicates that there is a line in the flow field where the maximum velocity is reached and beyond which the flow cannot continue. Tollmien in a subsequent paper (reference 5) called such lines limiting lines. The velocity at the limiting line is never subsonic. However, the true characteristics of such limiting lines and their significance were not investigated by Tollmien at that time. Recently Ringleb (reference 6) obtained another particular solution of isentropic irrotational flow in which the maximum velocity reached is approximately

twice the local sound velocity. For this flow also a limit-
ing line appeared beyond which the flow cannot continue.
Furthermore, he found the singular character of the limiting
line, that is, infinite acceleration and infinite pressure
gradient. Von Kármán (reference 7, particularly pp. 351-356)
demonstrated this fact for the general two-dimensional flow.
He also suggested that the limiting line is the envelope of
the Mach waves (fig. 1) and thus can occur only in a super-
sonic region. He also took its appearance as the criterion
for breakdown of isentropic irrotational flow. This general
two-dimensional theory was established later by both Ringleb
(reference 8) and Tollmien (reference 9). Tollmien corrected
some mistakes in Ringleb's paper and, in addition, showed
that the flow definitely cannot continue beyond the limiting
line. The later fact introduced a "forbidden region" in the
flow bounded by the limiting line. This physical absurdity
can be avoided only by relaxing the condition of irrotation-
ality. But, as stated previously, for non-viscous fluids,
the transition from a flow without vorticity to that with
vorticity can be accomplished only by shock waves, which at
the same time also cause an increase in the entropy.

However, before it can be concluded that the appearance
of a limiting line, or the envelope of Mach waves, is the
general condition for breakdown of isentropic irrotational
flow, it must be proved that the singular behavior of limit-
ing lines is general and not limited to two-dimensional flow.
This is the purpose of the present paper. First the property
of limiting line in axially symmetric flow will be investigated
in detail. Then the general three-dimensional problem will be
sketched. These investigations confirm the results of Ringleb,
Von Kármán, and Tollmien for these more general cases.

Therefore, by considering only the steady flow of non-
viscous fluids, the criterion for breakdown of isentropic
irrotational flow is the appearance of a limiting line. For
the actual motion of a solid body, however, the flow is neither
steady nor non-viscous. Small disturbances always occur and
almost all real fluids have appreciable viscosity. The small
disturbances in the flow introduce the question of stability.
In other words, the solution found for isentropic irrotational
flow may be unstable even before the appearance of the limit-
ing line, and tends to transform itself into a rotational flow
involving shock waves at the slightest disturbance. If this
is the case, the criterion concerns not the limiting line, but
the stability limit. This problem has yet to be solved.

The effect of viscosity will be limited to the boundary layer if the pressure along the surface in the flow direction never increases too rapidly. Then outside the boundary layer the flow is isentropic and irrotational. If the gradient of pressure is too large, the boundary layer will separate from the surface. However, at low velocities such separation only widens the wake of the body and changes the pressure distribution over the body. But if the boundary layer separates at a point where the velocity outside the boundary layer is supersonic, additional effects may appear. The flow outside the boundary layer in this case can be regarded approximately as that of a solid body not of the original contour but of a new contour including the "dead water" region created by the separation. It is then immediately clear that the ideal isentropic irrotational flow around this new contour may have a limiting line. Hence, the actual flow then must involve shock waves. In other words, the separation of the boundary layer in the supersonic region may induce a shock wave and thus extend its influence far beyond the region of separation. Furthermore, the steep adverse pressure gradient across a shock wave may accentuate the separation. This interaction between the separation and the shock wave is frequently observed in experiments.

The above considerations indicate the possibility of the breakdown of isentropic irrotational flow outside the boundary layer even before the appearance of the limiting line. Therefore, the Mach number of the undisturbed flow at which the limiting line appears may be called the "upper critical Mach number." On the other hand, since shock waves can occur only in supersonic flow, the Mach number of the undisturbed flow at which the local velocity reaches the velocity of sound may be called the "lower critical Mach number." The actual critical Mach number for the appearance of shock waves and the compressibility burble must lie between these two limits. By carefully designing the contour of the body to avoid the crowding together of Mach waves to form an envelope and to eliminate adverse pressure gradients along the surface of the body, the compressibility burble can be delayed.

AXIALLY SYMMETRIC FLOW

The solution of the exact differential equations for an axially symmetric isentropic irrotational flow was first given by Frankle (reference 10). The method was developed independ-

ently by Ferrari (reference 11). Their method applies par-
ticularly to the case of supersonic flow over a body of
revolution with pointed nose. In this case, the flow at
the nose can be approximated by the well-known solution
for a cone. From this solution, the differential equation
is solved step by step, using the net of characteristics
which are real for supersonic velocities. In the following
investigation, the chief concern is not the solution of the
partial differential equation but rather the occurrence and
the properties of the limiting line in an isentropic irrota-
tional flow. The general plan of attack is that of Tollmien
(reference 9). However, here the calculation is based on
the Legendre transformation of velocity potential instead of
the stream function.

If q is the magnitude of the velocity, a the corre-
sponding velocity of sound assuming an isentropic process,
p the pressure, and ρ the density of fluid, the Bernoulli
equation gives

$$\frac{\rho}{\rho_0} = \left(1 - \frac{\gamma-1}{2} \frac{q^2}{a_0^2}\right)^{\frac{1}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2} \frac{q^2}{a^2}\right)^{-\frac{1}{\gamma-1}} \tag{1}$$

$$\frac{a^2}{a_0^2} = 1 - \frac{\gamma-1}{2} \frac{q^2}{a_0^2} = \left(1 + \frac{\gamma-1}{2} \frac{q^2}{a^2}\right)^{-1} \tag{2}$$

$$\frac{p}{p_0} = \left(1 - \frac{\gamma-1}{2} \frac{q^2}{a_0^2}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2} \frac{q^2}{a^2}\right)^{-\frac{\gamma}{\gamma-1}} \tag{3}$$

In these equations, the subscript o denotes quantities
corresponding to $q = 0$, and γ is the ratio of specific
heats of the fluid. Let the axis of symmetry be the x-axis,
the distance normal to x-axis be denoted by y, and the
velocity components along these two directions be denoted
by u and v, respectively (fig. 2). The x-y plane is,
therefore, a meridian plane. Then the kinematical relations
of the flow are given by the vorticity equation

$$v_x - u_y = 0^* \tag{4}$$

*Throughout this paper, partial derivatives are denoted
by subscripts. Thus $v_x \equiv \frac{\partial v}{\partial x}$, $u_y \equiv \frac{\partial u}{\partial y}$

and the continuity equation

$$\frac{\partial}{\partial x}\left(y\,\frac{\rho}{\rho_0}\,u\right) + \frac{\partial}{\partial y}\left(y\,\frac{\rho}{\rho_0}\,v\right) = 0 \qquad (5)$$

Equations (1) to (5), together with the relation $q^2 = u^2 + v^2$, specify the flow completely.

To simplify the problem, a velocity potential φ defined as follows is introduced:

$$u = \varphi_x, \qquad v = \varphi_y \qquad (6)$$

Then equation (4) is identically satisfied and equation (5), together with equations (1) and (2), gives the equation for φ.

$$\left(1 - \frac{u^2}{a^2}\right)\varphi_{xx} - 2\,\frac{uv}{a^2}\,\varphi_{xy} + \left(1 - \frac{v^2}{a^2}\right)\varphi_{yy} + \frac{v}{y} = 0 \qquad (7)$$

The characteristics of this differential equation, to be called the characteristics in the physical plane, are given by $g(x, y) = 0$, where $g(x, y)$ is determined by the following equation

$$\left(1 - \frac{u^2}{a^2}\right)g_x{}^2 - 2\,\frac{uv}{a^2}\,g_x\,g_y + \left(1 - \frac{v^2}{a^2}\right)g_y{}^2 = 0 \qquad (8)$$

It can be easily seen from this equation that g is real only when $q > a$. Therefore, the characteristics are real only in supersonic regions of the flow.

The meaning of characteristics in the physical plane is immediately clear if one calculates the relation between the slope of a characteristic and the slope of a stream line in the meridian or x-y plane. By the definition of the function $g(x,y)$, the value of g is zero, or constant, along a characteristic. Therefore, by writing a quantity evaluated at a certain constant value of a parameter with that parameter as a subscript, the slope of the characteristic in the physical plane is

$$\left(\frac{dy}{dx}\right)_g = - \frac{g_x}{g_y} \tag{9}$$

Along a streamline, the stream function ψ defined by following equations is constant:

$$\psi_y = y \frac{\rho}{\rho_0} u , \qquad \psi_x = - y \frac{\rho}{\rho_0} v \tag{10}$$

Therefore, the slope of a streamline is

$$\left(\frac{dy}{dx}\right)_\psi = \frac{v}{u} \tag{11}$$

Equations (8), (9), and (11) give

$$\left(\frac{dy}{dx}\right)_g = \frac{-\frac{uv}{a^2} \pm \sqrt{\frac{q^2}{a^2} - 1}}{1 - \frac{u^2}{a^2}} \cdot \left\{\left(\frac{dy}{dx}\right)_\psi \pm \tan \beta\right\} \div \left\{1 \mp \left(\frac{dy}{dx}\right)_\psi \tan \beta\right\} \tag{12}$$

where β is the Mach angle given by $\beta = \sin^{-1} \frac{a}{q}$. Therefore, equation (12) shows that the characteristics in the physical plane are inclined to the streamlines by an angle equal to the Mach angle. Such lines are the wave fronts of infinitesimal disturbances and are called Mach waves. In other words, characteristics in physical planes are the Mach waves in that plane. There are two families of Mach waves inclined symmetrically with respect to each streamline.

If to each pair of values of u and v, there is one pair of values of x, y, then x and y can be considered as functions of u, v. In other words, instead of taking x and y as independent variables u, v can be used as independent variables. The plane with u and v as coordinates is called the "hodograph plane." An equation in the hodograph plane corresponding to equation (7) can be obtained by means of Legendre's transformation. By writing

$$\chi = ux + vy - \varphi \tag{13}$$

it is seen that

$$\chi_u = x, \qquad \chi_v = y \tag{14}$$

Then equation (7) can be written as

$$\left(1 - \frac{u^2}{a^2}\right)\chi_{vv} + 2\frac{uv}{a^2}\chi_{uv}$$

$$+ \left(1 - \frac{v^2}{a^2}\right)\chi_{uu} + \frac{v}{\chi_v}\left[\chi_{uu}\chi_{vv} - \chi_{uv}^2\right] = 0 \tag{15}$$

The characteristics of equation (15) are given by $f(u,v) = 0$ where f is the solution of the following differential equation

$$\left\{\left(1 - \frac{u^2}{a^2}\right) + \frac{v}{\chi_v}\chi_{uu}\right\}f_v^2 + 2\left(\frac{uv}{a^2} - \frac{v}{\chi_v}\chi_{uv}\right)f_u f_v$$

$$+ \left\{\left(1 - \frac{v^2}{a^2}\right) + \frac{v}{\chi_v}\chi_{vv}\right\}f_u^2 = 0 \tag{16}$$

Equation (16) shows that the characteristics in the hodograph plane depend upon the values of the derivatives of χ which must be obtained from equation (15). In other words, the characteristics in the hodograph plane change with the flow and are not a constant set of curves as are those in two-dimensional problems.

To obtain the relation between the characteristics in the physical plane and those in the hodograph plane, it is noticed that equation (9) can be rewritten as

$$(dy)_g : (dx)_g = - g_x : g_y \tag{17}$$

Then equation (8) is equivalent to

$$\left(1 - \frac{u^2}{a^2}\right)(dy)_g^{\;2} + 2\,\frac{uv}{a^2}\,(dy)_g\,(dx)_g + \left(1 - \frac{v^2}{a^2}\right)(dx)_g^{\;2} = 0 \quad (18)$$

However, in general, equation (14) gives the following relation between the differentials of x and y and those of u and v:

$$dx = \chi_{uu}\,du + \chi_{uv}\,dv$$
$$\tag{19}$$
$$dy = \chi_{uv}\,du + \chi_{vv}\,dv$$

By means of these relations, equation (18) can be transformed into an equation for $(du)_g$ and $(dv)_g$. This transformed equation can be simplified by using equation (15). The final relation is

$$(\chi_{uu}\chi_{vv} - \chi_{uv}^{\;2})\left[\left\{\left(1 - \frac{u^2}{a^2}\right) + \frac{v}{\chi_v}\chi_{uu}\right\}(du)_g^{\;2}\right.$$

$$\left. - 2\left(\frac{uv}{a^2} - \frac{v}{\chi_v}\chi_{uv}\right)(du)_g\,(dv)_g + \left\{\left(1 - \frac{v^2}{a^2}\right) + \frac{v}{\chi_v}\chi_{vv}\right\}(dv)_g^{\;2}\right] = 0$$
$$\tag{20}$$

Therefore, if the first factor of equation (20) is not zero, the variations $(du)_g$ and $(dv)_g$ along a characteristic in physical plane must satisfy the relation

$$\left\{\left(1 - \frac{u^2}{a^2}\right) + \frac{v}{\chi_v}\chi_{uu}\right\}(du)_g^{\;2} - 2\left\{\frac{uv}{a^2} - \frac{v}{\chi_v}\chi_{uv}\right\}(du)_g\,(dv)_g$$

$$+ \left\{\left(1 - \frac{v^2}{a^2}\right) + \frac{v}{\chi_v}\chi_{vv}\right\}(dv)_g^{\;2} = 0 \qquad (21)$$

This is the same relation for the variations $(du)_f$ and $(dv)_f$ along a characteristic in the hodograph plane as can be seen from equation (16) and the following relation obtained from the definition of f

$$(dv)_f : (du)_f = - f_u : f_v \qquad (22)$$

The transformed characteristics of the physical plane and the characteristics of the hodograph plane themselves satisfy then the same first order differential equation. Therefore, these two types of curves are the same. In other words, the characteristics of the hodograph plane are the representation of Mach waves in the $u-v$ plane.

THE LIMITING LINE

Equation (20) shows that if

$$\chi_{uu} \chi_{vv} - \chi_{uv}^2 = 0 \qquad (23)$$

then the transformed differential equation for the characteristics of the physical plane, or Mach waves, is satisfied. Therefore, if there is a line in the hodograph plane along which the values of the derivatives of χ are such that equation (23) is true, then this line when transferred to the physical plane will have its slope equal to that of one family of Mach waves. Such lines are called the limiting hodograph in $u-v$ plane and the limiting line in physical plane. Since Mach waves occur only in the supersonic regions it is then evident that the limiting line must appear in these regions. The significance of the adjective "limiting" will be made clear as other properties of such lines are investigated.

Now the question arises: Can the limiting hodograph be a characteristic in $u-v$ plane? Along a limiting hodograph, equation (23) gives

$$\left(\frac{dv}{du}\right)_1 = - \frac{\chi_{uuu} \chi_{vv} - 2 \chi_{uv} \chi_{uuv} + \chi_{uu} \chi_{uvv}}{\chi_{uuv} \chi_{vv} - 2 \chi_{uv} \chi_{uvv} + \chi_{uu} \chi_{vvv}} \qquad (24)$$

where the subscript l denotes the value along a limiting hodograph. Now the general differential equation for χ, equation (15), is true for the whole $u-v$ plane; therefore; the equation is still true by differentiating it with respect to u and v. The results can be simplified by using equation (15) itself and equation (23). Then at the limiting hodograph,

$$\left[\left(1-\frac{v^2}{a^2}\right)+\frac{v}{\chi_v}\chi_{vv}\right]\chi_{uuu}+2\left[\frac{uv}{a^2}-\frac{v}{\chi_v}\chi_{uv}\right]\chi_{uuv}+\left[\left(1-\frac{u^2}{a^2}\right)\right.$$

$$\left.+\frac{v}{\chi_v}\chi_{uu}\right]\chi_{uvv}=(\gamma+1)\frac{u}{a^2}\chi_{vv}-2\frac{v}{a^2}\chi_{uv}+(\gamma-1)\frac{u}{a^2}\chi_{uv} \quad (25a)$$

$$\left[\left(1-\frac{v^2}{a^2}\right)+\frac{v}{\chi_v}\chi_{vv}\right]\chi_{uuv}+2\left[\frac{uv}{a^2}-\frac{v}{\chi_v}\chi_{uv}\right]\chi_{uvv}+\left[\left(1-\frac{u^2}{a^2}\right)\right.$$

$$\left.+\frac{v}{\chi_v}\chi_{uu}\right]\chi_{vvv}=(\gamma-1)\frac{v}{a^2}\chi_{vv}-2\frac{u}{a^2}\chi_{uv}+(\gamma+1)\frac{v}{a^2}\chi_{uu} \quad (25b)$$

Equations (24), (25a), and (25b) are the only available equations involving no higher derivative than the third. On the other hand, the slope of a characteristic in the hodograph plane can be calculated by equation (22),

$$\left(\frac{dv}{du}\right)_f = -\frac{f_u}{f_v} \quad (26)$$

This equation together with equation (16) gives

$$\left\{\left(1-\frac{v^2}{a^2}\right)+\frac{v}{\chi_v}\chi_{vv}\right\}\left(\frac{dv}{du}\right)_f^2$$

$$-2\left\{\frac{uv}{a^2}-\frac{v}{\chi_v}\chi_{uv}\right\}\left(\frac{dv}{du}\right)_f+\left\{\left(1-\frac{u^2}{a^2}\right)+\frac{v}{\chi_v}\chi_{uu}\right\}=0 \quad (27)$$

Therefore, if the limiting hodograph is a characteristic, then $\left(\frac{dv}{du}\right)_l$ must satisfy equation (27). However, a simple calcula-

tion shows that it is not even possible to obtain a relation between $\left(\dfrac{dv}{du}\right)_l$ and other quantities not involving the third-order derivatives of χ. Hence, $\left(\dfrac{dv}{du}\right)_l$ does not satisfy equation (27). In other words, the limiting hodograph is not a characteristic. Transferred to the physical plane, this means that the limiting line is not a Mach wave. But as shown in previous paragraphs, the limiting line is everywhere tangent to one family of Mach waves. Consequently, the limiting line must be the envelope of a family of Mach waves. This property of the limiting line can be taken as its physical definition.

LIMITING HODOGRAPH AND THE STREAMLINES

At the limiting hodograph both equations (15) and (23) hold. By eliminating one of the second-order derivatives, say χ_{uu}, the following relation is obtained

$$(\chi_{vv})_l = \frac{-\dfrac{uv}{a^2} \pm \sqrt{\dfrac{q^2}{a^2} - 1}}{1 - \dfrac{u^2}{a^2}} (\chi_{uv})_l \qquad (28)$$

The sign before the radical in equation (28) can be either positive or negative, but not both. This relation will be used presently to show that the streamlines and one family of characteristics are tangent in the u-v plane.

From equation (10), the differential of the stream function can be calculated as

$$d\psi = - y \frac{\rho}{\rho_0} v \, dx + y \frac{\rho}{\rho_0} u \, dy \qquad (29)$$

In this equation, y can be replaced by χ_v according to equation (14) and the differentials dx and dy replaced by the differential du and dv according to equation (19). Then

$$d\psi = \chi_v \frac{\rho}{\rho_0} \left[\left(- v \chi_{uu} + u \chi_{uv} \right) du + \left(- v \chi_{uv} + u \chi_{vv} \right) dv \right] \quad (30)$$

Along a streamline, $d\psi = 0$; therefore the slope of the stream-line in the hodograph plane is given by

$$\left(\frac{dv}{du} \right)_\psi = \frac{v \chi_{uu} - u \chi_{uv}}{- v \chi_{uv} + u \chi_{vv}} \quad (31)$$

At the limiting hodograph, equation (23) holds; therefore, equation (31) together with equation (28) gives

$$\left(\frac{dv}{du} \right)_{\psi, l} = - \left(\frac{\chi_{uv}}{\chi_{vv}} \right)_l = \frac{1 - \frac{u^2}{a^2}}{\frac{uv}{a^2} \mp \sqrt{\frac{q^2}{a^2} - 1}} \quad (32)$$

where the sign before the radical can be either negative or positive corresponding to the sign in equation (28).

On the other hand, the slope of the characteristics in the hodograph plane is determined by equation (27). By solving for $\left(\frac{dv}{du} \right)_f$ and simplifying the result with the aid of equation (15),

$$\left(\frac{dv}{du} \right)_f = \frac{\frac{uv}{a^2} - \frac{v}{\chi_v} \chi_{uv} \pm \sqrt{\frac{q^2}{a^2} - 1}}{\left(1 - \frac{v^2}{a^2} \right) + \frac{v}{\chi_v} \chi_{vv}} \quad (33)$$

The sign before the radical is either positive or negative corresponding to the two families of characteristics. By using the positive sign in conjunction with the positive sign in equation (28), and similarly for the negative sign,

$$\left(\frac{dv}{du} \right)_{f, l} = \frac{1 - \frac{u^2}{a^2}}{\frac{uv}{a^2} \mp \sqrt{\frac{q^2}{a^2} - 1}} \quad (34)$$

Equations (32) and (34) show that the streamlines and one
family of characteristics are tangent to each other at the
limiting hodograph. This result is the same as that ob-
tained for two-dimensional flow. (See references 7, 8, and
9.) Those equations when compared with equation (12) for
the slope of Mach waves in the physical plane yields the
interesting result that the streamlines and one family of
characteristics at the limiting hodograph are perpendicular
to the corresponding Mach waves at the limiting line.

Since

$$\left(\frac{dv}{du}\right)_\psi = -\frac{\psi_u}{\psi_v} \tag{35}$$

Equation (32) gives the following equation which holds at
the limiting hodograph

$$\left(1 - \frac{v^2}{a^2}\right)(\psi_u)_\iota^2 + 2\frac{uv}{a^2}(\psi_u)_\iota (\psi_v)_\iota + \left(1 - \frac{u^2}{a^2}\right)(\psi_v)_\iota^2 = 0 \tag{36}$$

This equation can be reduced to more familiar form by intro-
ducing the polar coordinates in the u-v plane:

$$u = q\cos\theta, \qquad v = q\sin\theta$$

where θ is the angle between the velocity vector and u-axis.
Then equation (36) takes the form

$$(\psi_q)_\iota^2 + \left(\frac{1}{q^2} - \frac{1}{a^2}\right)(\psi_\theta)_\iota^2 = 0 \tag{37}$$

This can be regarded as the equivalent of equation (23) for
defining the limiting hodograph. A similar relation exists
for two-dimensional flow. (See references 7, 8, and 9.)

Along a streamline, the ratio between $(dv)_\psi$ and
$(du)_\psi$ is given by equation (31). By substituting this
ratio in equation (19), the differential $(dx)_\psi$ and
$(dy)_\psi$ along a streamline is given as

$$(dx)_\psi = \frac{u\left[\chi_{uu}\,\chi_{vv} - \chi_{uv}^{\,2}\right]}{- v\,\chi_{uv} + u\,\chi_{vv}}\,(du)_\psi$$

$$(dy)_\psi = \frac{v\left[\chi_{uu}\,\chi_{vv} - \chi_{uv}^{\,2}\right]}{- v\,\chi_{uv} + u\,\chi_{vv}}\,(du)_\psi \quad\Bigg\} \quad (38)$$

At the limiting line, equation (23) is satisfied. Then equation (38) shows that at the limiting line, the streamline has a singularity. Or, more plainly, $(dx)_\psi$ and $(dy)_\psi$ at these points are infinitesimals of higher order than $(du)_\psi$ and $(dv)_\psi$. By writing s for the distance measured along a streamline, equation (38) gives immediately

$$(u_s)_\psi = \frac{-v\,\chi_{uv} + u\,\chi_{vv}}{q\left[\chi_{uu}\,\chi_{vv} - \chi_{uv}^{\,2}\right]} \quad (39)$$

Similarly,

$$(v_s)_\psi = \frac{v\,\chi_{uu} - u\,\chi_{uv}}{q\left[\chi_{uu}\,\chi_{vv} - \chi_{uv}^{\,2}\right]} \quad (40)$$

Therefore, at the limiting line, the acceleration along a streamline is infinitely large. Furthermore, since the pressure gradient $(p_s)_\psi$ along a streamline is

$$(p_s)_\psi = -\rho\,q\,q_s = -\rho\left[u\,(u_s)_\psi + v\,(v_s)_\psi\right] \quad (41)$$

the pressure gradient at the limiting line is also infinitely large.

Such infinite acceleration and pressure gradient lead
one to suspect that the fluid is thrown back at the limiting
line. In other words, the streamlines are doubled back at
this line of singularity. To investigate whether this is
true, the character of the relation $\chi_{uu} \chi_{vv} - \chi_{uv}^2 = 0$
along a streamline has to be determined. If the derivative
of this expression along a streamline is not zero, then
$\chi_{uu} \chi_{vv} - \chi_{uv}^2$ has only a simple zero at the intersection
of the limiting line and the streamline. Consequently, the
differentials $(dx)_\psi$ and $(dy)_\psi$ will change sign by passing
through the limiting hodograph in u-v plane along a stream-
line. Hence, the streamlines will double back and form a
cusp at the limiting line. The derivative of $\chi_{uu} \chi_{vv} - \chi_{uv}^2$
along the streamline can be calculated with the aid of equa-
tion (30)

$$\left[\frac{d}{du}\left(\chi_{uu} \chi_{vv} - \chi_{uv}^2\right)\right]_l = \chi_{uuu}\ \chi_{vv} - 2\ \chi_{uv}\ \chi_{uuv} + \chi_{uu}\ \chi_{uvv}$$

$$+ \frac{v\ \chi_{uu} - u\ \chi_{uv}}{-v\ \chi_{uv} + u\ \chi_{vv}}\left\{\chi_{uuv}\ \chi_{vv} - 2\ \chi_{uv}\ \chi_{uvv} + \chi_{uu}\ \chi_{vvv}\right\} \qquad (42)$$

The expression on the right of equation (42) cannot be reduced
to zero by the available relations, which consist of equation
(23), equation (15), and differentiated forms of equation (15).
Therefore, the expression concerned generally has only a simple
zero at the limiting hodograph and the streamlines are doubled
back at the limiting line. It will be shown later that there
is no solution possible beyond the limiting line. Hence, the
name limiting line.

ENVELOPE OF CHARACTERISTICS IN HODOGRAPH PLANE AND

LINES OF CONSTANT VELOCITY IN PHYSICAL PLANE

Since the limiting line is the envelope of the Mach waves
in the physical plane, it is interesting to see whether there
is also an envelope for the characteristics in the hodograph
plane. The characteristics in the u-v plane are determined
by equation (26). The envelope to them can be found by elim-
inating $\left(\frac{dv}{du}\right)_f$ between equation (26) and the following equation

$$\left\{ \left(1 - \frac{v^2}{a^3}\right) + \frac{v}{\chi_v} \chi_{vv} \right\} \left(\frac{dv}{du}\right)_f - \left\{ \frac{uv}{a^2} - \frac{v}{\chi_v} \chi_{uv} \right\} = 0 \qquad (43)$$

which is obtained by equating to zero the partial derivative of equation (26) with respect to $\left(\frac{dv}{du}\right)_f$. The result can be simplified by equation (15), and then it becomes simply

$$1 - \frac{u^2 + v^2}{a^2} + \frac{u^2 v^2}{a^4} = \frac{u^2 v^2}{a^4} \qquad (44)$$

This is satisfied by either

$$a = 0 \qquad (45)$$

or

$$u^2 + v^2 = a^2 \qquad (46)$$

The first condition, equation (45), when substituted into equation (26) gives

$$\left(\frac{dv}{du}\right)_{f, a=o} = - \frac{\bar{u}}{v} \qquad (47)$$

which shows that the circle of maximum velocity corresponding to $a = 0$, is the envelope of the characteristics in hodograph plane. The second condition, equation (46), is the spurious solution, since generally the characteristic at $q = a$ is not a tangent to the circle $q = a$. Hence $a = 0$ is the only envelope.

The lines of constant velocity in the hodograph plane are simply circles. Therefore

$$\left(\frac{dv}{du}\right)_q = - \frac{u}{v} \qquad (48)$$

By means of this relation and equation (19), the slope of the lines of constant velocity is given as

$$\left(\frac{dy}{dx}\right)_q = \frac{v \chi_{uv} - u \chi_{vv}}{v \chi_{uu} - u \chi_{uv}} \qquad (49)$$

This equation together with equation (30) gives the following interesting relation

$$\left(\frac{dy}{dx}\right)_q = - \frac{1}{\left(\frac{dv}{du}\right)_\psi} \tag{50}$$

In other words, a line of constant velocity in the physical plane is perpendicular to the streamline in the hodograph plane at the corresponding points.

THE LOST SOLUTION

Throughout the previous calculation, the possibility of using the Legendre transformation is assumed. This requires that for each pair of values of u, v there is one and only one pair of values of x, y. However, it is not al- ways true. It is possible to have a number of points in the physical plane having the same value of u and v. If this is the case, then evidently it is impossible to solve for x and y from the pair of functions u = u (x,y), v = v (x,y). Mathematically, the situation is expressed by saying that the Jacobian $\partial(u,v)/\partial(x,y)$ vanishes in the physical plane. Or

$$u_x v_y - u_y v_x = 0 \tag{51}$$

However, this is also the condition for a functional relation between u and v; for example, v can be expressed as a function of u. In other words, u and v are not independ- ent. Hence if a solution is "lost" or not included in the family of solutions allowing Legendre transformation, then for that solution,

$$v = v(u) \tag{52}$$

It is seen that equation (51) is then identically statisfied.

By eliminating ρ from the continuity equation, there is obtained

$$\left(1 - \frac{u^2}{a^2}\right) u_x - \frac{uv}{a^2} \left(u_y + v_x\right) + \left(1 - \frac{v^2}{a^2}\right) v_y + \frac{v}{y} = 0 \tag{53}$$

This equation can be rewritten in the following form by using equation (52)

$$\left\{\left(1-\frac{u^2}{a^2}\right)-\frac{uv}{a^2}\frac{dv}{du}\right\}u_x+\left\{\left(1-\frac{v^2}{a^2}\right)\frac{dv}{du}-\frac{uv}{a^2}\right\}u_y+\frac{v}{y}=0 \qquad (54)$$

The vorticity equation, equation (4), can be expressed as

$$\frac{dv}{du}u_x-u_y=0 \qquad (55)$$

From equations (54) and (55), it is possible to solve for u_x and u_y. The result is

$$\left[\left(1-\frac{u^2}{a^2}\right)-2\frac{uv}{a^2}\frac{dv}{du}+\left(1-\frac{v^2}{a^2}\right)\left(\frac{dv}{du}\right)^2\right]u_x=-\frac{v}{y} \qquad (56a)$$

$$\left[\left(1-\frac{u^2}{a^2}\right)-2\frac{uv}{a^2}\frac{dv}{du}+\left(1-\frac{v^2}{a^2}\right)\left(\frac{dv}{du}\right)^2\right]u_y=-\frac{v}{y}\frac{dv}{du} \qquad (56b)$$

By differentiating the first of equation (56) with respect to y, the second with respect to x, the following relation is obtained by subtraction:

$$\frac{d^2v}{du^2}u_x+\frac{1}{y}=0 \qquad (57)$$

Therefore

$$\frac{dv}{du}=\frac{f(y)-x}{y} \qquad (58)$$

or

$$y=\frac{f(y)-x}{\dfrac{dv}{du}}$$

where $f(y)$ is an undetermined function of y. However, equation (55) shows that for lines of constant values of u where $du=u_x\,(dx)_u+u_y\,(dy)_u=0$,

$$\left(\frac{dy}{dx}\right)_u = - \frac{1}{\left(\frac{dv}{du}\right)_u} = \text{constant} \qquad (59)$$

Hence, lines of constant values of u and v are straight lines. This restriction reduces the function $f(y)$ in equation (58) to a numerical constant. Put $f(y) = K$, equation (58) is then

$$y = \frac{K - x}{\dfrac{dv}{du}} \qquad (60)$$

Therefore lines of constant value of u and v are radial lines passing through the point $x = K$. Thus the lost solution is nothing but the well-known solution for the flow over a conical surface.

From equation (59), it is seen that lines of constant velocity are perpendicular to the tangent of the u-v curve at the corresponding points. By substituting the value of $\frac{1}{y}$ from equation (57) into equation (56a), a relation between u and v is obtained:

$$v \frac{d^2 v}{du^2} - \left(1 - \frac{v^2}{a^2}\right)\left(\frac{dv}{du}\right)^2 + 2 \frac{uv}{a^2} \frac{dv}{du} - \left(1 - \frac{u^2}{a^2}\right) = 0 \qquad (61)$$

This is the differential equation for determining the hodograph representing the flow over a cone. Figure 3 shows the hodograph for a cone of $30°$ semivertex angle and with a velocity at the surface of cone equal to $0.35\ c$. The maximum velocity is c — that is, the value of q corresponding to $a = 0$. Figure 3 is drawn from data given by Taylor and Maccoll (reference 12).

It may well be mentioned here that the lost solution for the axially symmetric flow is not limited to supersonic velocity as is the case for two-dimensional flow. In fact, Taylor and Maccoll show that for small forward velocity of the cone, supersonic velocities occur only just after the head shock wave. The velocity decreases as the surface of the cone is approached. Finally, it becomes subsonic for points near the surface of the cone. Figure 4 shows a few examples taken from

their calculations (reference 12). The dotted curves in the
figure are the Mach waves. The dotted straight lines are
the boundaries between the supersonic and the subsonic regions.
Furthermore, spark photographs of a conical shell in actual
flight taken by Maccoll (reference 13) do not indicate the
presence of shock waves in regions of flow where such transi-
tion from supersonic to subsonic velocities is expected.
Therefore, at least for this particular type of flow, a
smooth transition through sonic velocity actually takes
place.

CONTINUATION OF SOLUTION BEYOND THE LIMITING LINE

Since it is shown in a previous paragraph that the
streamlines are generally turned back at the limiting line,
the question arises: Is it possible to continue the solution
beyond the limiting line? Of course, there are two ways of
continuing the solution: The new solution is joined either
smoothly to the given solution at the limiting line or with
a discontinuity. As shown before, the limiting line is the
envelope of one family of the Mach waves. Then at every
point of this line its direction differs from that of the
streamline by an angle equal to the Mach angle. But the
Mach angle is not zero except at points where the velocity
of fluid has reached the maximum velocity and the ratio
$\frac{a}{q} = 0$. Therefore, the limiting line generally does not
coincide with the streamline, and the discontinuity at the
junction of the solution at the limiting line cannot be that
of a vortex sheet. The only other type of discontinuity is
the shock wave. However, the angle between the limiting line
and the flow direction is equal to the Mach angle. Then ac-
cording to the result of the theory of shock waves, the dis-
continuity across such a line vanishes. In other words,
there cannot be a discontinuity at the limiting line. There-
fore, it is impossible to join a new solution at the limit-
ing line with a discontinuity.

As to the second possibility of joining a new solution
smoothly at the limiting line, it is seen that the flow beyond
the limiting line must be irrotational and isentropic since
the limiting line cannot be a shock wave. There are only two
types of isentropic irrotational flow; namely, one that allows
the Legendre transformation, and one that does not, the "lost
solution." Investigate the second alternative first. If the

solution beyond the limiting line belongs to the so-called lost solution, then since the junction at the limiting line must be smooth, the values of u and v at the limiting line must also satisfy equation (61). But the slope $\left(\dfrac{dv}{du}\right)_l$ at the limiting line is given by equation (24). The second derivative $\left(\dfrac{d^2v}{du^2}\right)_l$ will then involve the fourth order derivatives of χ. Besides these expressions, the available relations are equations (15), (23), (25a), (25b), and three more equations obtained by differentiating equations (25) with respect to u and v. However, it is still impossible for $\left(\dfrac{dv}{du}\right)_l$ to satisfy an equation like equation (61) where no derivative of χ appears. Hence, the limiting hodograph does not satisfy the equation for the lost solution. In other words, the lost solution cannot be used to continue the flow beyond the limiting line.

The only remaining possibility is to continue the flow smoothly by another solution obtainable by a Legendre transformation. Smooth continuation means that the values of u, v, and ρ must be the same at the junction, the limiting line. Since shock waves do not appear, isentropic relations still hold. The density ρ is determined by velocity only. The value of u and v are determined by the coordinates in the hodograph plane. The position of the limiting line in the physical plane is determined by χ_u, χ_v. Therefore, the problem can be stated as follows: At a certain given curve $u(\lambda)$, $v(\lambda)$ in the hodograph plane, the limiting hodograph, the values of χ_u, χ_v are given; λ is the parameter along the given curve. It is required to determine a new solution of the differential equation (equation (15)) with these initial values. First of all, it is seen that with the given data, the left-hand sides of the following equations are given:

$$\frac{d}{d\lambda}(\chi_u) = \chi_{uu}\frac{du}{d\lambda} + \chi_{uv}\frac{dv}{d\lambda} \qquad (62a)$$

$$\frac{d}{d\lambda}(\chi_v) = \chi_{uv}\frac{du}{d\lambda} + \chi_{vv}\frac{dv}{d\lambda} \qquad (62b)$$

Therefore

$$\chi_{uv} = \left[-\frac{dv}{d\lambda} \chi_{vv} + \frac{d}{d\lambda}(\chi_v) \right] \Big/ \frac{du}{d\lambda} \qquad (63a)$$

$$\chi_{uu} = \left[\left(\frac{dv}{d\lambda}\right)^2 \chi_{vv} - \frac{dv}{d\lambda}\frac{d}{d\lambda}(\chi_v) + \frac{du}{d\lambda}\frac{d}{d\lambda}(\chi_u) \right] \Big/ \left(\frac{du}{d\lambda}\right)^2 \qquad (63b)$$

By substituting those values into equation (15), the second-degree terms reduce to

$$\chi_{uu}\chi_{vv} - \chi_{uv}{}^2 = \left[\frac{dv}{d\lambda}\frac{d}{d\lambda}(\chi_v) + \frac{du}{d\lambda}\frac{d}{d\lambda}(\chi_u) \right] \chi_{vv} \Big/ \left(\frac{du}{d\lambda}\right)^2$$

$$+ \left[\frac{d}{d\lambda}(\chi_v) \right]^2 \Big/ \left(\frac{du}{d\lambda}\right)^2 \qquad (64)$$

which is linear in χ_{vv}. Therefore χ_{vv} can be uniquely determined by equation (15). In other words, with the given data, the second order derivatives of χ at the given curve $u(\lambda)$, $v(\lambda)$ can be determined uniquely, in spite of the fact that the differential equation (15) is of second degree. Friedrichs and Lewy (reference 13) have shown that under these circumstances, the function χ within a region R (fig. 5) bounded by two characteristics and the given curve is uniquely determined except for an additional constant. Consequently there can be only one solution corresponding to the given data at the limiting hodograph. However, this solution is the very one which gives the reverse flow at limiting line. Therefore, it is impossible to continue the solution beyond the limiting line even by a Legendre transformation.

Since all three alternatives fail to offer a way of continuing the solution, the limiting line is truly an impossible boundary to cross. In other words, the region beyond the limiting line is a forbidden region. This physical absurdity can be resolved only by the breakdown of isentropic irrotational flow.

GENERAL THREE-DIMENSIONAL FLOW

The methods used in previous sections for investigating the axially symmetric flow can be easily extended to the general three-dimensional case. In the present section, this investigation will be sketched briefly and the results indicated.

Let the three components of velocity along the three coordinate axes x, y, and z be denoted by u, v, and w, respectively. Then by introducing a velocity potential φ defined by

$$u = \varphi_x, \qquad v = \varphi_y, \qquad w = \varphi_z \qquad (65)$$

the differential equation for φ of an isentropic irrotational flow can be written as (reference 7)

$$a^2 (\varphi_{xx} + \varphi_{yy} + \varphi_{zz}) =$$

$$u^2 \varphi_{xx} + v^2 \varphi_{yy} + w^2 \varphi_{zz} + 2vw\varphi_{yz} + 2wu\varphi_{zx} + 2uv\varphi_{xy} \qquad (66)$$

If, for every triad of u, v, w, there is only one triad of x, y, z, then the Legendre transformation can be used. Thus

$$\chi = ux + vy + wz - \varphi \qquad (67)$$

and

$$\chi_u = x, \qquad \chi_v = y, \qquad \chi_w = z \qquad (68)$$

The differential equation for φ, equation (66), is then transformed into

$$a^2 \left[BC - F^2 + CA - G^2 + AB - H^2 \right] = u^2 (BC - F^2) + v^2 (CA - G^2)$$
$$+ w^2 (AB - H^2) + 2vw (GH - AF) + 2wu (HF - BG) + 2uv (FG - CH) \qquad (69)$$

where the following notations are used

$$A = \chi_{uu}, \quad B = \chi_{vv}, \quad C = \chi_{ww}, \quad F = \chi_{vw}, \quad G = \chi_{wu}, \quad H = \chi_{uv} \quad (70)$$

By analogy with the axially symmetric case, the limiting hodograph surface is defined as the surface in the u, v, w space, or hodograph space, where the following relation holds:

$$\Delta = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0 \qquad (71)$$

The properties of this limiting hodograph and the corresponding limiting surface can be found by considering the behavior of streamlines and characteristics at such surfaces.

From equation (68) the differentials of x, y, and z can be written as

$$dx = A\, du + H\, dv + G\, dw \qquad (72a)$$

$$dy = H\, du + B\, dv + F\, dw \qquad (72b)$$

$$dz = G\, du + F\, dv + C\, dw \qquad (72c)$$

Along a streamline, the differentials dx, dy, and dz must be proportional to u, v, and w, respectively. Thus the equation of a streamline in physical space is

$$\frac{(dx)_\psi}{u} = \frac{(dy)_\psi}{v} = \frac{(dz)_\psi}{w} \qquad (73)$$

where the subscript ψ indicates values taken along the streamline. The equation of a streamline in hodograph space

is obtained by eliminating dx,   dy,   and   dz   from equation
(73) by equation (72). The result is

$$\frac{(du)_\psi}{\overline{a}u + \overline{h}v + \overline{g}w} = \frac{(dv)_\psi}{\overline{h}u + \overline{b}v + \overline{f}w} = \frac{(dw)_\psi}{\overline{g}u + \overline{f}v + \overline{c}w} \qquad (74)$$

where $\overline{\overline{a}}$ is the co-factor of A in the determinant   Δ   of
equation (71), $\overline{b}$ the co-factor of B, and so forth. Equa-
tion (74) can be used, in turn, to eliminate two of the three
differentials du, dv, and dw in the right of equation
(72). The result is

$$(dx)_\psi = \frac{u\Delta\,du}{\overline{a}u + \overline{h}v + \overline{g}\,w} \qquad (75a)$$

$$(dy)_\psi = \frac{v\Delta\,dv}{\overline{h}u + \overline{b}v + \overline{f}w} \qquad (75b)$$

$$(dz)_\psi = \frac{w\Delta\,dw}{\overline{g}u + \overline{f}v + \overline{c}w} \qquad (75c)$$

At the limiting surface, Δ = 0 as defined by equation (71);
therefore the streamlines have a singularity there. Similar
to the axially symmetric flow, the streamlines generally are
turned back and form a cusp at this surface. The accelera-
tion and the pressure gradient are, of course, infinitely
large at such places.

The characteristic surface g(x,y,z) = 0 in physical
space is determined by the equation

$$a^2 \left[g_x{}^2 + g_y{}^2 + g_z{}^2 \right] =$$

$$u^2 g_x{}^2 + v^2 g_y{}^2 + w^2 g_z{}^2 + 2vw g_y g_z + 2wu g_z g_x + 2uv g_x g_y \qquad (76)$$

Since this equation is a second-degree equation, there are two families of surfaces passing through each point. These surfaces are the wave fronts of infinitesimal disturbances in the flow and can be called the Mach surfaces. The characteristic surface $f(u, v, w) = 0$ in the hodograph space is determined by the equation

$$a^2 \left[(B+C)f_u^2 + (C+A)f_v^2 + (A+B)f_w^2 - 2F \, f_u f_w - 2G \, f_w f_u - 2H \, f_u f_v \right]$$

$$= u^2 \left[C \, f_v^2 + B \, f_w^2 - 2F \, f_v f_w \right] + v^2 \left[C \, f_u^2 + A \, f_w^2 - 2G \, f_w \, f_u \right]$$

$$+ w^2 \left[B \, f_u^2 + A \, f_v^2 - 2H \, f_u f_v \right] + 2vw \left[H \, f_w f_u + G \, f_u f_v \right.$$

$$\left. - F \, f_u^2 - A \, f_v \, f_w \right] + 2wu \left[H \, f_v f_w + F \, f_u f_v - G \, f_v^2 - B \, f_w f_u \right]$$

$$+ 2uv \left[G \, f_w f_v + F \, f_w f_u - H \, f_w^2 - C \, f_u \, f_v \right] \tag{77}$$

By transforming equation (76) for Mach surfaces to hodograph space, it can be shown that the transformed equation is satisfied either by the characteristics in hodograph space determined by equation (77) or by the limiting hodograph determined by equation (71). Therefore, here again the limiting surface is the envelope of a family of Mach surfaces.

By using equations (74) and (77), it is possible to show that the streamlines in the hodograph space are tangent to the characteristic surfaces at the limiting hodograph. Furthermore, by using equations (69), (71), and (74), the inclination of the streamlines at the limiting hodograph can be calculated. In fact, if $(ds)^2 = (du)^2 + (dv)^2 + (dw)^2$, $q^2 = u^2 + v^2 + w^2$, the following relation is obtained

$$\left(\frac{ds}{dq} \right)_{\psi, l} = \frac{q}{a} \quad \text{or} \quad - \frac{q}{a} \tag{78}$$

This relation is really equivalent to equation (32). In other words, at the limiting hodograph, the inclination of

the streamlines and characteristics to the q = constant
surface is equal to the Mach angle (fig. 6). It thus seems
the breakdown of general steady isentropic irrotational flow
of nonviscous fluid is connected with the appearance of the
envelope of Mach waves in physical space and the tangency of
streamlines and characteristics in hodograph space.

California Institute of Technology,
      Pasadena, Calif., August 24, 1943.

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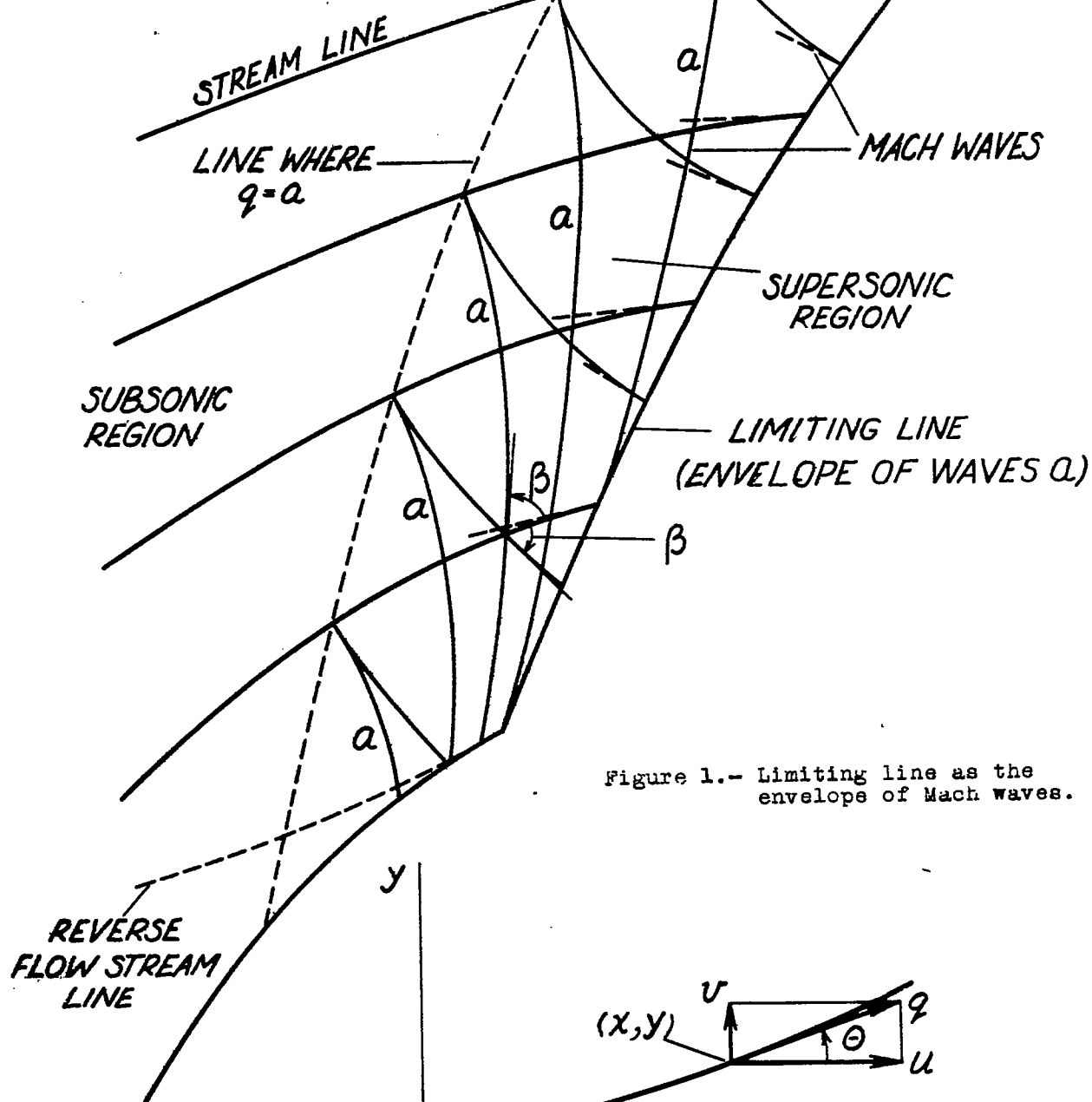   Variablen. Math. Annalen, vol. 99, 1928, pp. 200-
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STREAM LINE

LINE WHERE
q=a

MACH WAVES

a

SUPERSONIC
REGION

SUBSONIC
REGION

a

a

β

LIMITING LINE
(ENVELOPE OF WAVES a)

β

a

a

REVERSE
FLOW STREAM
LINE

Figure 1.— Limiting line as the
envelope of Mach waves.

y

v

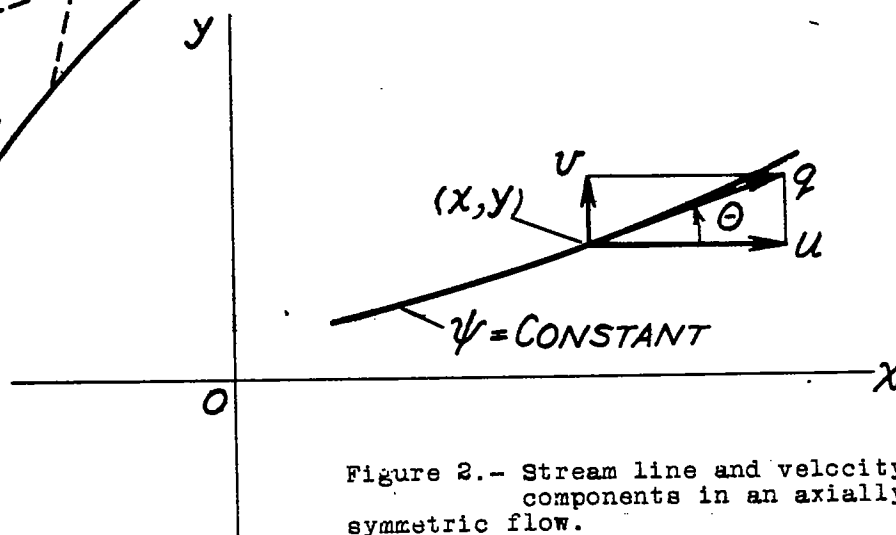(x,y)

q

Θ

u

$\psi = CONSTANT$

O

x

Figure 2.— Stream line and velocity
components in an axially
symmetric flow.
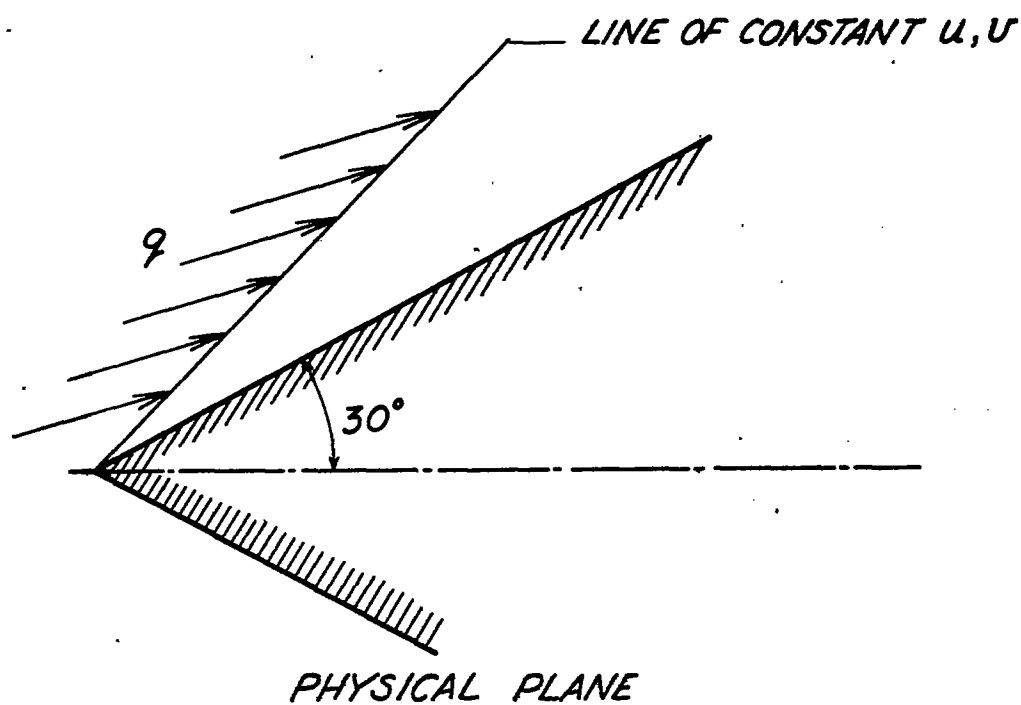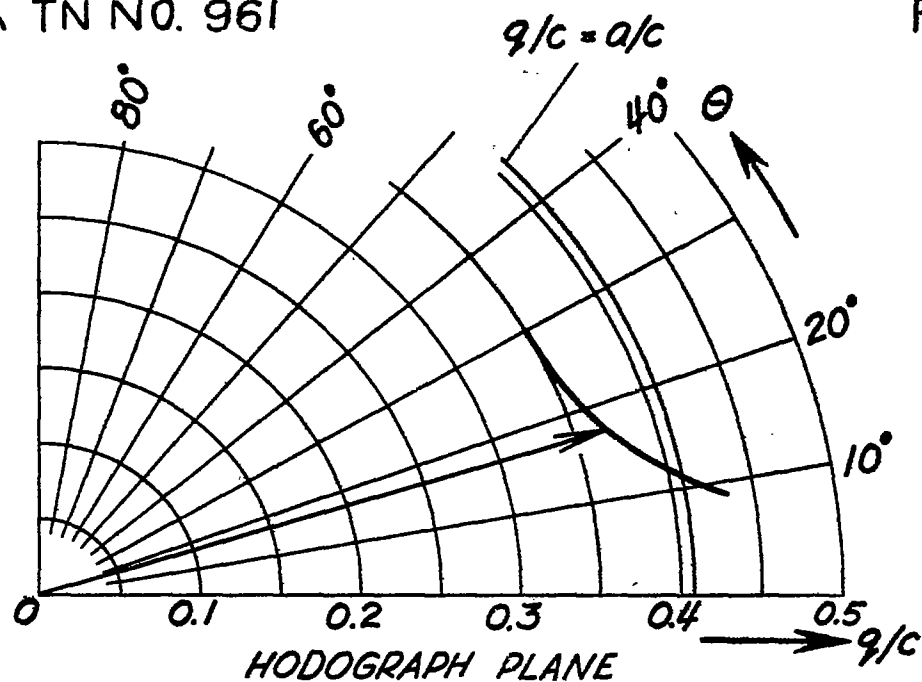
Figure 3.— Hodograph of the flow over a cone of 30° half vertex angle
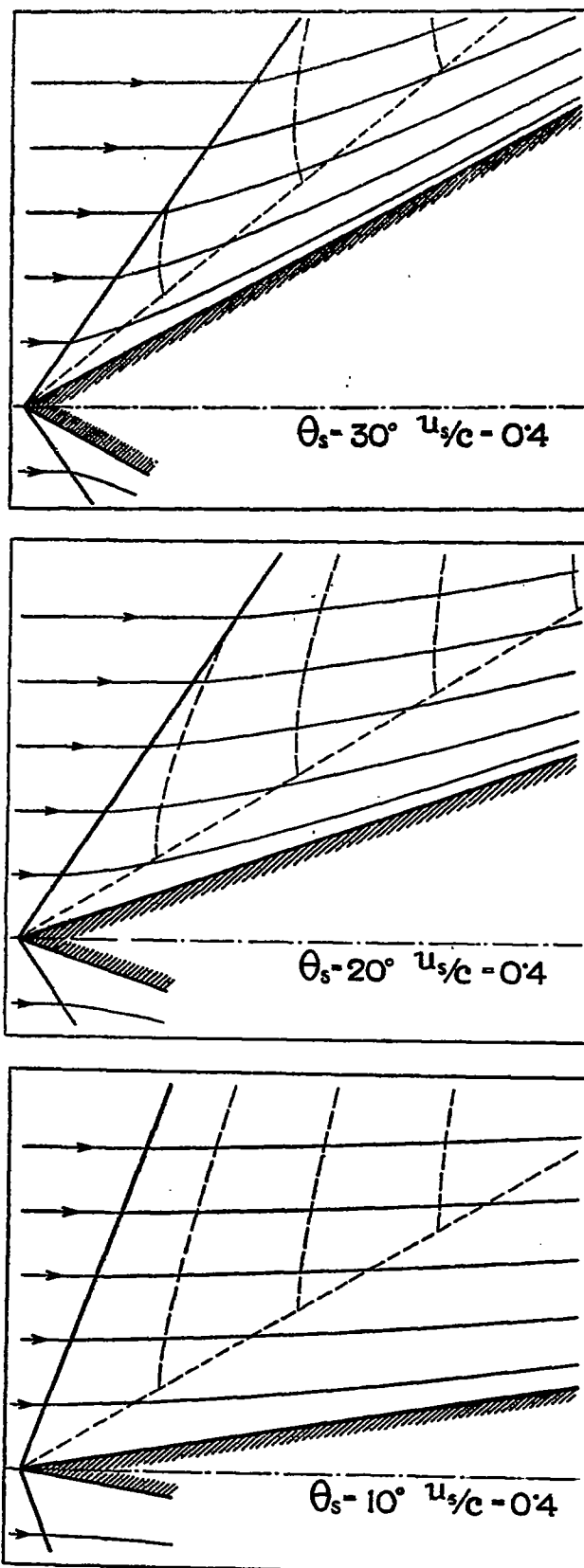and a surface velocity q equal to 0.350.

Figure 4.- Flow over cones of various vertex angles involving subsonic regions. θ_S = half vertex angle, u_S = velocity over the surface of cone.
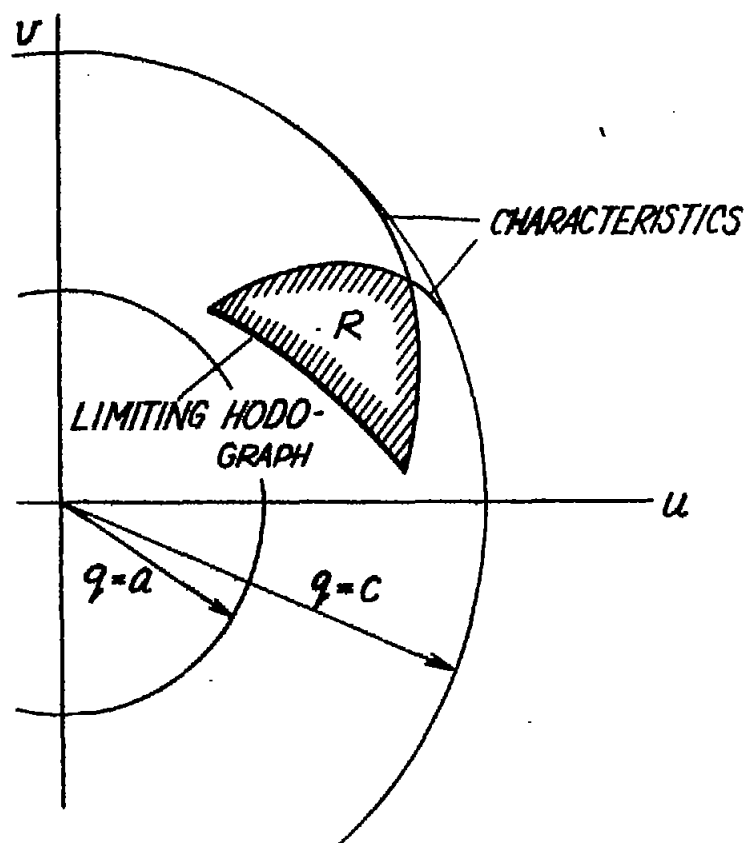
Figure 5.- Region R where the solution is uniquely
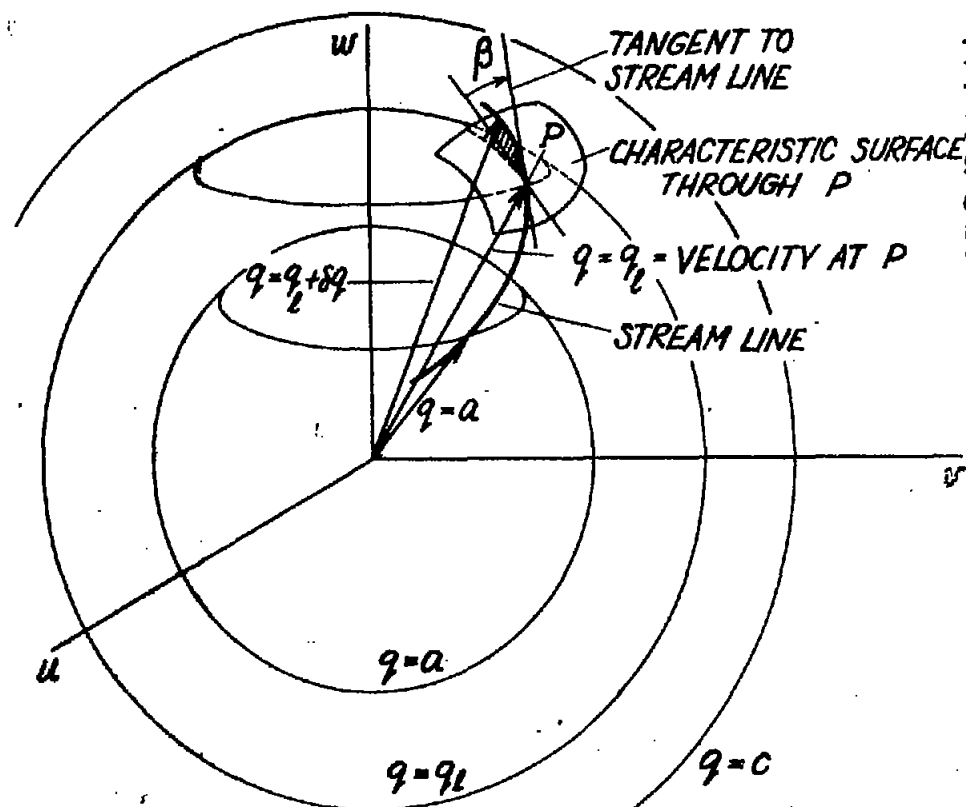determined by given data at the limiting
hodograph.

Figure 6.- Geometrical relations between stream line
and characteristic surface at limiting
surface in hodograph space.